Generalized traveling waves on complete Riemannian manifolds

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Abstract

In the article of H. Berestycki and F. Hamel, *On a general definition of transition waves*, there is a generalization of the classical definition of a transition wave in Euclidean spaces (e.g. a travelling wave or an invasive front) to the case where the level sets of the wave are no longer planes but surfaces. We will prove that the same results and properties on general transition waves that appear in the cited article hold in the case of a non-compact complete Riemannian manifold, namely: (1) the wave is associated to a generalized front, which moves "close" to the level sets of the wave; (2) there is a mean propagation speed of the wave, which is independent of the choice of the associated front; (3) in the case of an invasion the wave is an increasing function in time.

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1 Definition of general travelling waves on manifolds

1.1 Complete Riemannian manifolds

Let $\mathcal{M}$ be a n-dimensional, C^∞ Riemannian manifold, and let $g_{ij}(\xi)$ be its metric in the local coordinates $\xi \in \mathbb{R}^n$. We will suppose that $\mathcal{M}$ is closed, connected, without boundary, unbounded (i.e. non-compact) and complete. Recall that a manifold $\mathcal{M}$ is complete if any of the following statements hold (see Bishop and Crittenden [4], Theorem 5, p. 154):

1. $\mathcal{M}$ is complete as a metric space, i.e. any Cauchy sequence in $\mathcal{M}$ has a limit in $\mathcal{M}$.
2. All bounded closed subsets of $\mathcal{M}$ are compact.
3. All geodesics are infinitely extendible.

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These conditions are equivalent, and they imply the next result:

($*$) Any $x, y \in \mathcal{M}$ can be joined by a geodesic whose arc length equals the geodesic distance $d(x, y)$.

The geodesic distance is defined as:

$$d(x,y) := \inf\{\, |\gamma| : \gamma \in G \,\}\,,$$

where G is the set of all continuous and piecewise C^∞ curves from x to y, and $|\gamma|$ is the arc length of a curve γ. Moreover, the geodesic distance is a continuous functions, and the topology it generates is equivalent to the topology of $\mathcal{M}$ as a manifold (see Bishop and Crittenden [4], pp. 124-125).

1.2 Reaction-diffusion equations on manifolds

From now on, we will work on a manifold $\mathcal{M}$ with the properties stated in Section 1.1.

Consider the scalar reaction-diffusion equation

$$\begin{cases} \partial_t u = D\Delta_{\mathcal{M}} u + F(t,x,u)\,; & t \in \mathbb{R},\ x \in \mathcal{M}\,, \\ u(0,x) = u_0(x)\,; & x \in \mathcal{M}\,. \end{cases} \tag{1}$$

$\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator, which in local coordinates takes the form (note that we use the sum convention)

$$\Delta_{\mathcal{M}} u = \frac{1}{\sqrt{g}}\,\partial_j[\,\sqrt{g}\,g^{ij}\,\partial_i u\,]\,, \tag{2}$$

where $\partial_i := \partial_{\xi_i}$, $(g^{ij}) = (g_{ij})^{-1}$ and $g = det(g_{ij})$. Another way of expressing the Laplace-Beltrami operator is

$$\begin{aligned} \Delta_{\mathcal{M}} u &= g^{ij}\partial_{ij} u + \frac{1}{\sqrt{g}}\partial_j[\,g^{ij}\sqrt{g}\,]\partial_i u \\ &= g^{ij}\partial_{ij} u - g^{ij}\Gamma^k_{ij}\partial_k u\,, \end{aligned}$$

where $\partial_{ij} = \partial^2_{\xi_i\xi_j}$ and

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}[\partial_j g^{il} + \partial_i g^{jl} - \partial_l g^{ij}]$$

are the Schwartz-Christoffel symbols.

The assumptions on the nonlinearity $F(t,x,u)$ are:

- Either F is C^1 and both F and $\partial_u F$ are globally bounded, or
- Either F is bounded, continuous in (t,x) and locally Lipschitz continuous in u, uniformly in (t,x).

We can also suppose that F does not depend on the variables (t,x), i.e. $F = F(u)$.

1.3 Fronts, waves and invasions

For any two subsets $A, B \subset \mathcal{M}$ denote

$$d(A,B) := \inf\{\, d(x,y) \,:\, x \in A,\; y \in B\}\,.$$

Definition 1 Generalized profile
A generalized profile is a family $(\Omega_t^{\pm}, \Gamma_t,)_{t\in\mathbb{R}}$ of subsets of $\mathcal{M}$ with the following properties:

1. *Ω_t^- and Ω_t^+ are non-empty disjoint subsets of $\mathcal{M}$, for any $t \in \mathbb{R}$.*
2. *$\Gamma_t = \partial\Omega_t^- \cap \partial\Omega_t^+$ and $\mathcal{M} = \Gamma_t \cup \Omega_t^- \cup \Omega_t^+$, for any $t \in \mathbb{R}$.*
3. *$\sup\{\, d(x,\Gamma_t) \,:\, t \in \mathbb{R},\, x \in \Omega_t^-\} = \sup\{\, d(x,\Gamma_t) \,:\, t \in \mathbb{R},\, x \in \Omega_t^+\} = +\infty$*

Suppose that there exist $p^-, p^+ \in \mathbb{R}$ such that $F(t,x,p^{\pm}) = 0$ for all $t \in \mathbb{R}$ and all $x \in \mathcal{M}$. Then $u \equiv p^{\pm}$ are solutions of (1).

Definition 2 Generalized front
Let $u(t,x)$ be a time-global classical solution of (1) such that $u \not\equiv p^{\pm}$. Then $u(x,t)$ is a generalized front between p^- and p^+ if there exists a generalized front $(\Omega_t^{\pm}, \Gamma_t,)_{t\in\mathbb{R}}$ such that

$$|u(t,x) - p^{\pm}| \to 0 \text{ uniformly when } x \in \Omega_t^{\pm} \text{ and } d(x,\Gamma_t) \to +\infty.$$

Observe that the generalized profile $(\Omega_t^{\pm}, \Gamma_t,)_{t\in\mathbb{R}}$ is not uniquely determined. However, it is important to bear in mind that any generalized front $u(t,x)$ is by definition associated to a certain generalized profile.

Definition 3 Generalized invasion
Let $u(t,x)$ be a generalized front. We say that p^+ invades p^- or that $u(t,x)$ is a generalized invasion of p^- by p^+ (resp. p^- invades p^+ or that $u(t,x)$ is a generalized invasion of p^+ by p^-) if

(i) $\Omega_t^+ \subset \Omega_s^+$ (resp. $\Omega_s^- \subset \Omega_t^-$) for all $t \leq s$.

(ii) $d(\Gamma_t, \Gamma_s) \to +\infty$ when $|t-s| \to +\infty$.

Remark that if p^+ invades p^- (resp. p^- invades p^+) then $u(t,x) \to p^{\pm}$ when $t \to \pm\infty$ (resp. when $t \to \mp\infty$), locally uniformly in $\mathcal{M}$ with respect to the geodesic distance $d(\cdot,\cdot)$.

Example 1 *Let us consider the reaction diffusion equation (1) in $\mathcal{M} = \mathbb{R}^n$ with $D = 1$. In this case the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ is the classical Laplacian $\Delta = \partial_{ii}$, and (1) takes the form*

$$\begin{cases} \partial_t u = \Delta u + F(u)\,; & t \in \mathbb{R},\; x \in \mathbb{R}^n\,, \\ u(0,x) = u_0(x)\,; & x \in \mathbb{R}^n\,. \end{cases} \qquad (3)$$

If we are looking for generalized travelling waves of the form $u(t,x) = \phi(x \cdot e - ct)$, with $e \in \mathbb{R}^n$ is a unit vector and $c > 0$, the natural choice for the generalized front is

$$\Gamma_t = \{x \in \mathbb{R}^n \,:\, x \cdot e - ct = 0\}\,, \quad \Omega_t^{\pm} = \{x \in \mathbb{R}^n \,:\, \pm(x \cdot e - ct) > 0\}\,.$$

In this framework, (1) becomes

$$c\phi' + \phi'' + F(\phi) = 0\,.$$

Example 2 *Consider the parametrization*

$$\begin{array}{rcl} S^1 \times \mathbb{R} & \longrightarrow & \mathcal{M} \subset \mathbb{R}^3 \\ (\xi, z) & \longmapsto & (x, y, z) = (r\cos\xi\,, \sin\xi, z) \end{array}$$

where $\xi \in [0, 2\pi)$ *and* $r > 0$ *is fixed. The Laplace-Beltrami operator is*

$$\Delta_{\mathcal{M}} = \frac{1}{r^2}\partial_{\xi\xi} + \partial_{zz}\,.$$

If we want to study travelling waves in the axial direction z*, it is a natural guess to consider* $u(t, x) = \phi(z - ct)$*. In this case (1) takes the same form as in Example 1, i.e.*

$$c\phi' + \phi'' + F(\phi) = 0\,,$$

and the corresponding front is

$$\Gamma_t = \{(x, y, z) \in \mathcal{M}\,:\, z - ct = 0\}\,, \quad \Omega_t^{\pm} = \{(x, y, z) \in \mathcal{M}\,:\, \pm(z - ct) > 0\}\,.$$

From now on, for the sake of simplicity, we will call a *profile* any generalized profile, a *front* any generalized front, and a *invasion* any generalized invasion.

Definition 4 Global mean speed
A front $u(t, x)$ *has global mean speed* $c > 0$ *if the profile* $(\Omega_t^{\pm}, \Gamma_t,)_{t\in\mathbb{R}}$ *is such that*

$$\frac{d(\Gamma_t, \Gamma_s)}{|t - s|} \to c \quad \text{when } |t - s| \to +\infty.$$

In Example 1 we have

$$\Gamma_t = \{x \in \mathbb{R}^n\,:\, x \cdot e - ct = 0\}\,,$$

which implies that

$$d(\Gamma_t, \Gamma_s) = c|t - s|\,.$$

In consequence, the velocity of propagation of the profile $(\Omega_t^{\pm}, \Gamma_t,)_{t\in\mathbb{R}}$ coincides with its global mean speed.

Remark 1 *All these definitions hold also in the case of a reaction-diffusion equation or system on a growing manifold* $\mathcal{M}_t$ *(see Labadie [7] and Plaza* et al *[10] for the definition and properties of reaction-diffusion equations on growing manifolds).*

2 Properties of fronts on manifolds

For any $x \in \mathcal{M}$ and any $r > 0$ define

$$\begin{array}{rcl} B(x, r) & = & \{\, y \in \mathcal{M}\,:\, d(x, y) \le r\,\}\,, \\ S(x, r) & = & \{\, y \in \mathcal{M}\,:\, d(x, y) = r\,\}\,. \end{array}$$

Theorem 1 Level sets
Let $p^- < p^+$ and suppose that $u(t,x)$ is a time-global solution of (1) such that

$$p^- < u(t,x) < p^+ \quad \text{for all } (t,x) \in \mathbb{R} \times \mathcal{M}.$$

1. *Suppose $u(t,x)$ is front between p^- and p^+ (or between p^+ and p^-) with the following properties:*
   - *(a) There exists $\tau > 0$ such that $\sup\{\, d(x, \Gamma_{t-\tau}) \,:\, t \in \mathbb{R},\, x \in \Gamma_t \,\} < +\infty$, and*
   - *(b) $\sup\{\, d(y, \Gamma_t) \,:\, y \in \overline{\Omega_t^{\pm}} \cap S(x,r) \,\} \to +\infty$ when $r \to +\infty$, uniformly in $t \in \mathbb{R}$ and $x \in \Gamma_t$.*

   Then:
   - *(i) $\sup\{\, d(x, \Gamma_t) \,:\, u(t,x) = \lambda \,\} < +\infty$ for all $\lambda \in (p^-, p^+)$.*
   - *(ii) $p^- < \inf\{\, u(t,x) \,:\, d(x,\Gamma_t) \le C \,\} \le \sup\{\, u(t,x) \,:\, d(x,\Gamma_t) \le C \,\} < p^+$ for all $C \ge 0$.*

2. *Conversely, if (i) and (ii) hold for a certain profile $(\Omega_t^{\pm}, \Gamma_t,)_{t\in\mathbb{R}}$ and there exists $d_0 > 0$ such that for all $d \ge d_0$ the sets*
$$\{\, (t,x) \in \mathbb{R} \times \mathcal{M} \,:\, x \in \overline{\Omega_t^{\pm}},\, d(x,\Gamma_t) \ge d \,\}$$
are connected, then $u(t,x)$ is a front between p^- and p^+ (or between p^+ and p^-).

Theorem 2 Uniqueness of the global mean speed
Let $p^- < p^+$ and suppose that $u(t,x)$ is a front between p^- and p^+, where its associated profile $(\Omega_t^{\pm}, \Gamma_t,)_{t\in\mathbb{R}}$ satisfies (b) in Theorem 1. If $u(t,x)$ has a global mean speed $c > 0$ then it is independent of the profile. In other words, if for any other profile $(\tilde{\Omega}_t^{\pm}, \tilde{\Gamma}_t,)_{t\in\mathbb{R}}$ satisfying (b) the front $u(t,x)$ has global mean speed $\tilde{c}$, then $\tilde{c} = c$.

Theorem 3 Monotonicity
Let $p^- < p^+$ and suppose $F(t,x,u)$ satisfies the following conditions:

- *(α) $s \mapsto F(s,x,u)$ is non-decreasing for all $(x,u) \in \mathbb{M} \times \mathbb{R}$.*
- *(β) There exists $\delta > 0$ such that $q \mapsto F(t,x,q)$ is non-increasing for all $q \in \mathbb{R} \setminus (p^- + \delta, p^+ - \delta)$.*

Let $u(t,x)$ be a invasion of p^- by p^+ and assume (as in Theorem 1) that:

- *(a) There exists $\tau > 0$ such that $\sup\{\, d(x, \Gamma_{t-\tau}) \,:\, t \in \mathbb{R},\, x \in \Gamma_t \,\} < +\infty$, and*
- *(b) $\sup\{\, d(y, \Gamma_t) \,:\, y \in \overline{\Omega_t^{\pm}} \cap S(x,r) \,\} \to +\infty$ uniformly in $t \in \mathbb{R}$ and $x \in \Gamma_t$ when $r \to +\infty$.*

Then:

1. *$p^- < u(t,x) < p^+$ for all $(t,x) \in \mathbb{R} \times \mathcal{M}$.*
2. *$u(t,x)$ is increasing in time, i.e. $u(t+s,x) > u(t,x)$ for all $s > 0$.*

3 Proofs

3.1 Proof of Theorem 1

Proposition 1 $\sup\{\, d(x,\Gamma_t)\,:\,u(t,x)=\lambda\,\}<+\infty$ *for all* $\lambda\in(p^-,p^+)$.

Proof: Suppose it is not true. Then there is a $\lambda\in(p^-,p^+)$ and a sequence $(t_n,x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathcal{M}$ such that $u(t_n,x_n)=\lambda$ and $d(x_n,\Gamma_{t_n})\to+\infty$ when $n\to+\infty$.

Up to extraction of a subsequence, either $x_n\in\overline{\Omega^-_{t_n}}$ for all n, or either $x_n\in\overline{\Omega^+_{t_n}}$ for all n. Since $u(t,x)$ is a generalized wave then in the first case $u(t_n,x_n)\to p^-$, whereas in the second case $u(t_n,x_n)\to p^+$, hence in any case we reach a contradiction. □

Proposition 2 $p^-<\inf\{\,u(t,x)\,:\,d(x,\Gamma_t)\le C\,\}\le\sup\{\,u(t,x)\,:\,d(x,\Gamma_t)\le C\,\}<p^+$ *for all* $C\ge 0$.

Proof: Suppose it is not true. Then there exist $C>0$ and a sequence $(t_n,x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathcal{M}$ such that $d(x_n,\Gamma_{t_n})\le C$ and $u(t_n,x_n)\to p^-$ or p^+ when $n\to+\infty$. Since both cases can be treated with similarly let us only prove the case

$$u(t_n,x_n)\to p^-\quad\text{when}\quad n\to+\infty\,. \tag{4}$$

Since $d(x_n,\Gamma_{t_n})\le C$ for all n, Property (a) implies that there exist $\tau>0$ and a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ such that $\tilde{x}_n\in\Gamma_{t_n-\tau}$ for all n and

$$\sup\{\,d(x_n,\tilde{x}_n)\,:\,n\in\mathbb{N}\,\}<+\infty\,.$$

From the definition of a front it follows that there exists $d>0$ such that

$$d(y,\Gamma_t)\ge d\quad\text{implies that}\quad u(t,y)\ge\frac{p^-+p^+}{2}\quad\text{for all } (t,y)\in\mathbb{R}\times\overline{\Omega^+_t}.$$

From Property (b) there exists $r>0$ such that, for each n, there exists $y_n\in\overline{\Omega^+_{t_n-\tau}}$ satisfying

$$d(\tilde{x}_n,y_n)=r\quad\text{and}\quad d(y_n,\Gamma_{t_n-\tau})\ge d\,.$$

Therefore

$$u(t_n-\tau,y_n)\ge\frac{p^-+p^+}{2}\quad\text{for all } n. \tag{5}$$

On the other hand, the function $v(t,x):=u-p^+$ is non-negative. Moreover, recalling $F(t,x,p^-)=0$ it follows that v satisfies the nonlinear equation

$$\partial_t v=D\Delta_{\mathcal{M}}v+F(t,x,u)\quad\text{in } \mathbb{R}\times\mathcal{M}.$$

This implies that v satisfies the linear equation

$$\partial_t v=D\Delta_{\mathcal{M}}v+b(t,x)v\quad\text{in } \mathbb{R}\times\mathcal{M},$$

where

$$b(t,x)=\begin{cases}\frac{F(t,x,u(t,x))-F(t,x,p^-)}{u(t,x)-p^-}=\frac{F(t,x,u(t,x))}{u(t,x)-p^-}, & \text{if } u(t,x)\ne p^-,\\[2ex] \Theta, & \text{if } u(t,x)=p^-,\end{cases}$$

- $\Theta=\partial_u F(t,x,p^-)$ if F is C^1 in u, uniformly in (t,x), and both F and $\partial_u F$ are globally bounded, or

- $\Theta = K$, if $F(t,x,\cdot)$ is Lipschitz continuous and bounded in u, uniformly in (t,x).

In both cases $b(t,x)$ is at least bounded and measurable. Therefore we can apply Harnack's inequality (Lemma 1) to $v(t,x)$ and obtain that there exists $C_1 > 0$ such that

$$u(t_n-\tau,y_n)-p^- = v(t_n-\tau,y_n) \le C_1 v(t_n,x_n) = C_1[u(t_n,x_n)-p^-]\,.$$

It is important to remark that C_1 depends on τ but is independent of n. In consequence, we can take the limit $n \to +\infty$ and use (4)-(5) to obtain

$$\frac{p^- - p^+}{2} \le 0\,,$$

which is a contradiction. □

We suppose here that Propositions 1 and 2 hold, and that there exists $d_0 > 0$ such that for all $d \ge d_0$ the sets

$$\{\,(t,x)\in\mathbb{R}\times\mathcal{M}\,:\,x\in\overline{\Omega_t^{\pm}},\,d(x,\Gamma_t)\ge d\,\}$$

are connected.

Define

$$\begin{aligned} m^- &:= \liminf\{\,u(t,x)\,:\,x\in\overline{\Omega_t^-},\,d(x,\Gamma_t)\to+\infty\,\}\,,\\ M^- &:= \limsup\{\,u(t,x)\,:\,x\in\overline{\Omega_t^-},\,d(x,\Gamma_t)\to+\infty\,\}\,. \end{aligned}$$

We affirm that $m^- = M^-$. Indeed, if $m^- < M^-$ then $\lambda := (m^- + M^-)/2 \in (m^-, M^-)$. Moreover, by hypothesis $p^- \le m^- \le M^- \le p^+$, which implies that $\lambda \in (p^-, p^+)$. Therefore, using Property (i) it follows that there exists $C > 0$ such that

$$d(x,\Gamma_t) < C \ \text{ for all } \ (t,x)\in\mathbb{R}\times\mathcal{M} \ \text{ with } \ u(t,x)=\lambda.$$

On the other hand, by definition of lim inf and lim sup there exists two points $(t_1,x_1),(t_2,x_2)\in\mathbb{R}\times\overline{\Omega_t^-}$ such that

$$u(t_1,x_1) < \lambda < u(t_2,x_2) \ \text{ and } \ d(x_i,\Gamma_{t_i}) \ge \max\{C,d_0\} \ \text{ for } \ i=1,2.$$

Now recall that by hypothesis the set

$$\{\,(t,x)\in\mathbb{R}\times\mathcal{M}\,:\,x\in\overline{\Omega_t^-},\,d(x,\Gamma_t)\ge\max\{C,d_0\}\,\}$$

is connected. Therefore, since $u(t,x)$ is continuous there exist

$$(t,x)\in\mathbb{R}\times\overline{\Omega_t^-} \ \text{ such that } \ d(x,\Gamma_t)\ge\max\{C,d_0\} \ \text{ and } \ u(t,x)=\lambda,$$

which contradicts the definition of C_0.

Therefore $m^- = M^-$, and in consequence $u(t,x)$ has a limit, i.e.

$$u(t,x)\to m^- \ \text{ uniformly in } \ x\in\overline{\Omega_t^-} \ \text{ when } \ d(x,\Gamma_t)\to+\infty\,.$$

Similarly, switching all $-$ signs by $+$ we obtain that

$$u(t,x) \to m^+ \text{ uniformly in } x \in \overline{\Omega_t^+} \text{ when } d(x,\Gamma_t) \to +\infty\,.$$

We now affirm that $p^- = \min\{m^-, m^+\}$ and $p^+ = \max\{m^-, m^+\}$. Indeed, if $p^- < \min\{m^-, m^+\}$ then there exist $\varepsilon > 0$ and $C > 0$ such that

$$u(t,x) \geq p^- + \varepsilon > p^- \text{ for all } (t,x) \text{ with } d(x,\Gamma_t) \geq C.$$

But by Property (ii) we also have that

$$\inf\{\, u(t,x) \,:\, d(x,\Gamma_t) \leq C \,\} > p^-\,.$$

In consequence,

$$\inf\{\, u(t,x) \,:\, (t,x) \in \mathbb{R} \times \mathcal{M} \,\} > p^-\,,$$

which contradicts the fact that the range of $u(t,x)$ is the whole interval (p^-, p^+).

In conclusion, $p^- = \min\{m^-, m^+\}$, and analogously we can show that $p^+ = \max\{m^-, m^+\}$.

Finally, note that if $m^- = p^-$ and $m^+ = p^+$ then $u(t,x)$ is a front between p^- and p^+, whereas if $m^- = p^+$ and $m^+ = p^-$ then $u(t,x)$ is a front between p^+ and p^-.

This concludes the proof of Theorem 1.

3.2 Proof of Theorem 2

Let $p^- < p^+$ and suppose that the front $u(t,x)$ has global mean speed $c > 0$ with respect to the profile $(\Omega_t^\pm, \Gamma_t,)_{t\in\mathbb{R}}$. Let $(\tilde{\Omega}_t^\pm, \tilde{\Gamma}_t,)_{t\in\mathbb{R}}$ be another profile for $u(t,x)$ satisfying (b) in Theorem 1. We have to prove that $u(t,x)$ has also a global mean speed with respect to the new front $(\tilde{\Omega}_t^\pm, \tilde{\Gamma}_t,)_{t\in\mathbb{R}}$, and that it is precisely c.

Proposition 3 *There exists $C > 0$ such that*

$$d(x,\tilde{\Gamma}_t) \leq C \text{ for all } t \in \mathbb{R} \text{ and all } x \in \Gamma_t\,. \tag{6}$$

Proof: If (6) does not hold then there is a sequence $(t_n, x_n)_{n\in\mathbb{N}}$ such that

$$x_n \in \Gamma_{t_n} \text{ and } d(x_n, \tilde{\Gamma}_{t_n}) \to +\infty \text{ when } n \to +\infty\,. \tag{7}$$

Up to extraction of a subsequence, either $x_n \in \tilde{\Omega}_{t_n}^-$ for all n, or either $x_n \in \tilde{\Omega}_{t_n}^+$ for all n. Since both cases can be proven similarly, we will suppose that $x_n \in \tilde{\Omega}_{t_n}^-$ for all n.

Since $u(t,x)$ is a front then there exists $A > 0$ such that

$$|u(t,x) - p^+| \leq \frac{p^+ - p^-}{2} \text{ for all } (t,x) \in \mathbb{R} \times \overline{\Omega_t^+} \text{ with } d(x,\Gamma_t) \geq A. \tag{8}$$

Property (b) implies that for each n there exists $r > 0$ and $y_n \in \overline{\Omega^+_{t_n}}$ such such that

$$d(x_n, y_n) = r \ \text{ and } \ d(y_n, \Gamma_{t_n}) \geq A \ . \tag{9}$$

Note that the uniformity of the limit in (b) implies that $r > 0$ is independent of n. Therefore, using (7) and (9) it follows that

$$d(y_n, \tilde{\Gamma}_{t_n}) \to +\infty \ \text{ when } \ n \to +\infty \ .$$

Moreover, if n is sufficiently big then $y_n \in \tilde{\Omega}^-_{t_n}$ because, should $y_n \in \overline{\tilde{\Omega}^+_{t_n}}$, then using $\tilde{\Gamma}_{t_n} = \partial\tilde{\Omega}^-_{t_n} \cap \partial\tilde{\Omega}^+_{t_n}$ we would obtain

$$d(x_n, \tilde{\Gamma}_{t_n}) \leq d(x_n, \overline{\tilde{\Omega}^+_{t_n}}) \leq d(x_n, y_n) = r \, ,$$

which contradicts (7).

In the light of this we have that $u(t_n, y_n) \to p^-$ when $n \to +\infty$. But we also have that $y_n \in \overline{\Omega^+_{t_n}}$, so using (8)-(9) we obtain

$$|u(t_n, y_n) - p^+| \leq \frac{p^+ - p^-}{2} \quad \text{for all } n.$$

In consequence, making $n \to +\infty$ it follows that

$$|p^- - p^+| \leq \frac{p^+ - p^-}{2} \, ,$$

which is a contradiction. In conclusion, (6) holds. $\square$

Now let $\varepsilon > 0$. Then for any two times $t, s \in \mathbb{R}$:

- There exist $x \in \Gamma_t$ and $y \in \Gamma_s$ such that $d(x, y) \leq d(\Gamma_t, \Gamma_s) + \varepsilon$.
- There exist $\tilde{x} \in \tilde{\Gamma}_t$ and $\tilde{y} \in \tilde{\Gamma}_s$ such that $d(\tilde{x}, \tilde{y}) \leq d(\tilde{\Gamma}_t, \tilde{\Gamma}_s) + \varepsilon$.
- $d(x, \tilde{x}) \leq d(x, \tilde{\Gamma}_t) + \varepsilon$ and $d(y, \tilde{y}) \leq d(y, \tilde{\Gamma}_s) + \varepsilon$.

Therefore, by virtue of (6) we have

$$\begin{aligned} d(\tilde{x}, \tilde{y}) &\leq d(\tilde{x}, x) + d(x, y) + d(y, \tilde{y}) \\ &\leq (C + \varepsilon) + (d(\Gamma_t, \Gamma_s) + \varepsilon) + (C + \varepsilon) \\ &\leq d(\Gamma_t, \Gamma_s) + 2C + 3\varepsilon \, . \end{aligned}$$

and in consequence

$$d(\tilde{\Gamma}_t, \tilde{\Gamma}_s) \leq d(\Gamma_t, \Gamma_s) + 2C + 3\varepsilon \, .$$

Since $\varepsilon > 0$ was arbitrary then

$$d(\tilde{\Gamma}_t, \tilde{\Gamma}_s) \leq d(\Gamma_t, \Gamma_s) + 2C \ \text{ for all } \ t, s \in \mathbb{R},$$

which implies that

$$\limsup_{|t-s| \to +\infty} \frac{d(\tilde{\Gamma}_t, \tilde{\Gamma}_s)}{|t - s|} \leq \limsup_{|t-s| \to +\infty} \frac{d(\Gamma_t, \Gamma_s)}{|t - s|} = c \, . \tag{10}$$

Now observe that interchanging the roles of the sets $\Omega_t^{\pm}$ and $\tilde{\Omega}_t^{\pm}$ it can be shown that there exists $\tilde{C} > 0$ such that

$$d(\tilde{x}, \Gamma_t) \le \tilde{C} \text{ for all } t \in \mathbb{R} \text{ and all } \tilde{x} \in \tilde{\Gamma}_t \ ,$$

i.e. the "tilde" version of (6). In the light of this, it can be shown as well that

$$d(\Gamma_t, \Gamma_s) \le d(\tilde{\Gamma}_t, \tilde{\Gamma}_s) + 2\tilde{C} \text{ for all } t, s \in \mathbb{R},$$

and in consequence

$$c = \liminf_{|t-s|\to+\infty} \frac{d(\Gamma_t, \Gamma_s)}{|t-s|} \le \liminf_{|t-s|\to+\infty} \frac{d(\tilde{\Gamma}_t, \tilde{\Gamma}_s)}{|t-s|} \,. \tag{11}$$

From (10) and (11) we deduce that the limit

$$\lim_{|t-s|\to+\infty} \frac{d(\tilde{\Gamma}_t, \tilde{\Gamma}_s)}{|t-s|}$$

exists and is equal to c.

This concludes the proof of Theorem 2.

3.3 Proof of Theorem 3

Proposition 4 *$p^- < u(t,x) < p^+$ for all $(t,x) \in \mathbb{R} \times \mathcal{M}$.*

Proof: Define

$$m := \inf\{\, u(t,x) - p^- \ : \ (t,x) \in \mathbb{R} \times \mathcal{M} \,\}$$

and suppose $m < -\delta < 0$. Let $(t_n, x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R} \times \mathcal{M}$ such that

$$u(t_n, x_n) - p^- \to m \quad \text{when } n \to \infty \,.$$

Since $u(t,x)$ is a front and

$$u(t_n, x_n) \to m + p^- < p^- < p^+$$

we have that the sequence $d(x_n, \Gamma_{t_n})$ is bounded (otherwise $u(t_n, x_n)$ would converge to either p^- or p^+ by definition). From Property (a) it follows that, for any $n \in \mathbb{N}$, there exists a point $\tilde{x}_n \in \Gamma_{t_n-\tau}$ such that the sequence $d(x_n, \tilde{x}_n)_{n\in\mathbb{N}}$ is bounded. Since $u(t,x)$ is a front there exists $d > 0$ such that $u(t,x) \ge p^-$ whenever $d(x, \Gamma_t) \ge d$.

From Property (b) there is a sequence $(y_n)_{n\in\mathbb{N}}$ such that, for any $n \in \mathbb{N}$:

- $y_n \in \overline{\Omega^+_{t_n-\tau}}$,
- $d(y_n, \tilde{x}_n) = r$,
- $d(y_n, \Gamma_{t_n-\tau}) \ge d$.

Using this properties we have that

$$u(t_n - \tau, y_n) \geq p^- \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Define

$$w(t,x) := u(t,x) - p^- + m \geq 0\,.$$

If $n \in \mathbb{N}$ is big enough then it follows from condition (β) that w satisfies

$$\partial_t w \geq D\Delta_{\mathcal{M}} w + F(t,x,u) \geq D\Delta_{\mathcal{M}} w + F(t,x,w) \quad \text{for all } t \geq t_n.$$

Therefore, using the same argument of Proposition 2 we can show that there exists a function $b \in L^\infty(\mathbb{R} \times \mathcal{M})$ such that w satisfies the linear equation

$$\partial_t w \geq D\Delta_{\mathcal{M}} w + b(t,x) w \quad \text{in } \mathbb{R} \times \mathcal{M}.$$

As before, we apply Harnack's inequality (Lemma 1) to $w(t,x)$ and obtain that there exists $C_1 > 0$, independent of n, such that

$$w(t_n - \tau, y_n) - p^- = u(t_n - \tau, y_n) - p^- - m \leq C_1 v(t_n, x_n) = C_1[u(t_n, x_n) - p^- - m]\,.$$

Making $n \to +\infty$ and using (4)-(5) it follows that the left-hand side converges to

$$p^+ - p^- - m > 0\,,$$

whereas the right-hand side converges by definition to zero. In consequence, we have obtained a contradiction.

We have thus shown that $m \geq 0$, which implies that $p^- < u(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathcal{M}$. The inequality $u(t,x) < p^+$ can be proven with the same arguments. $\square$

Remark 2 *Decreasing $\delta > 0$ if necessary we can assume that $2\delta < p^+ - p^-$. Moreover, from Definition 2 there exists $A > 0$ such that for all $(t,x) \in \mathbb{R} \times \mathcal{M}$:*

- *If $x \in \overline{\Omega_t^-}$ and $d(x, \Gamma_t) \geq A$ then $u(t,x) \leq p^- + \delta$.*
- *If $x \in \overline{\Omega_t^+}$ and $d(x, \Gamma_t) \geq A$ then $u(t,x) \geq p^+ - \delta/2$.*

Since p^+ invades p^- there exists $s_0 > 0$ such that

$$\Omega_{t+s}^+ \subset \Omega_t^+ \text{ and } d(\Gamma_{t+s}, \Gamma_t) \geq 2A \text{ for all } t \in \mathbb{R} \text{ and all } s \geq s_0. \tag{13}$$

Let $t \in \mathbb{R}$, $s \geq s_0$ and $x \in \overline{\Omega}$ be fixed. On the one hand, if $x \in \overline{\Omega_t^+}$ then (13) implies that $x \in \overline{\Omega_{t+s}^+}$ and $d(, \Gamma_{t+s}) \geq 2A$ because any continuous path from x to $\overline{\Gamma_{t+s}}$ meets $\overline{\Gamma_t}$. On the other hand, if $x \in \overline{\Omega_t^-}$ and $d(, \Gamma_t) \leq 2A$ then using (13) it follows that $d(, \Gamma_{t+s}) \geq A$ and $x \in \overline{\Omega_{t+s}^+}$. In both cases we obtain that

$$u^s(t,x) := u(t+s, x) \geq p^+ - \delta/2 \geq p^+ - \delta\,.$$

Proposition 5 *Define*

$$\omega_A^- := \{ (t,x) \in \mathbb{R} \times \mathcal{M} \,:\, x \in \overline{\Omega_t^-} \text{ and } d(x,\Gamma_t) \geq A \} .$$

Then for all $s \geq s_0$ we have

$$u^s(t,x) \geq u(t,x) \text{ for all } (t,x) \in \omega_A^-.$$

Proof: Fix $s \geq s_0$ and define

$$\varepsilon^* := \inf\{ \varepsilon > 0 \,:\, u^s \geq u - \varepsilon \quad \text{in } \omega_A^- \} \tag{14}$$

Since $u(t,x)$ is bounded it follows that $\varepsilon^* \geq 0$ is well defined.

We claim that $\varepsilon^* = 0$. Indeed, let us suppose that $\varepsilon^* > 0$. Then there exists a sequence

$$0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$$

such that $\varepsilon_n < \varepsilon^*$ for all $n \in \mathbb{N}$ and $\varepsilon_n \to \varepsilon^*$. Moreover, using (14) we have, for each $n \in \mathbb{N}$, a point $(t_n, x_n) \in \omega_A^-$ such that

$$u(t_n + s, x_n) < u(t_n, x_n) - \varepsilon_n .$$

Observe that the sequence $d(x_n, \Gamma_{t_n})$ is bounded. If it is not, we can find a subsequence such that $d(x_n, \Gamma_{t_n}) \to +\infty$, which implies that

$$u(t_n, x_n) - p^- \to 0 . \tag{15}$$

But on the other hand we have

$$u(t_n, x_n) - p^- > u(t_n + s, x_n) + \varepsilon_n - p^- \geq \varepsilon_n ,$$

and in consequence

$$u(t_n, x_n) - p^- \to \varepsilon^* > 0$$

which contradicts (15).

Since $d(x_n, \Gamma_{t_n})$ is bounded, from assumption (a) there exists a sequence of points $\tilde{x}_n \in \mathcal{M}$ such that $\tilde{x}_n \in \Gamma_{t_n - \tau}$ and

$$\sup\{ d(x_n, \tilde{x}_n) \,:\, n \in \mathbb{N} \} < +\infty .$$

By hypothesis p^+ invades p^-, which implies that $\overline{\Omega_{t_n - t}^-} \supset \Omega_{t_n}^-$ for all $t \geq 0$. In consequence, using that $x_n \in \overline{\Omega_{t_n}^-}$ and $d(x_n, \Gamma_{t_n}) \geq A$ it follows that $x_n \in \overline{\Omega_{t_n - \tau}^-}$ and $d(x_n, \Gamma_{t_n - \tau}) \geq A$ for all $n \in \mathbb{N}$. Therefore, we can find a sequence $y_n \in \overline{\Omega_{t_n - \tau}^-}$ such that

$$A = d(y_n, \Gamma_{t_n - \tau}) = d(x_n, \Gamma_{t_n - \tau}) - d(x_n, y_n) .$$

Hence

$$\begin{aligned} d(x_n, y_n) &= d(x_n, \Gamma_{t_n - \tau}) - A \\ &\leq d(x_n, \tilde{x}_n) - A < +\infty . \end{aligned}$$

Since the sequence $d(x_n, y_n)$ is bounded we can construct a sequence of continuous paths in $\mathcal{M}$

$$P_n := \gamma_n([0,1])\,, \quad \gamma_n : [0,1] \to \overline{\Omega^-_{t_n-\tau}}$$

such that $\gamma_n(0) = x_n$ and $\gamma_n(1) = y_n$ for all $n \in \mathbb{N}$. Moreover, the completeness of the manifold $\mathcal{M}$ alows us to choose the paths P_n such as their length is precisely $d(x_n, y_n)$ and

$$d(\gamma_n(\sigma), \Gamma_{t_n-\tau}) \geq A \quad \text{for all } \sigma \in [0,1].$$

In consequence, using Remark 2 we can infer that

$$u(t_n - t, \gamma_n(\sigma)) \leq p^- + \delta \quad \text{for all } \sigma \in [0,1],\, n \in \mathbb{N} \text{ and } t \geq 0.$$

In the light of this we obtain that

$$u(t_n - \tau, x_n) - \varepsilon^* < u(t_n - \tau, x_n) \leq p^- + \delta\,.$$

Moreover, from Lemma 3 we have that $u(t,x)$ is uniformly continuous, which implies that there exists a small ρ independent of $n \in \mathbb{N}$ such that

$$u(t,x) - \varepsilon^* < p^- + \delta$$

for all $(t,x) \in \mathbb{R} \times \mathcal{M}$ satisfying

$$t \in (t_n - \tau - \rho, t_n - \tau + \rho) \quad \text{and} \quad d(x, x_n) < \rho\,.$$

Let us focus in the region where $u - \varepsilon^* \leq p^- + \delta$. On the one hand, since $F(t,x,\cdot)$ is non-increasing in this region it follows that

$$\begin{aligned} \partial_t(u - \varepsilon^*) &= D\Delta(u - \varepsilon^*) + F(t,x,u) \\ &\leq D\Delta(u - \varepsilon^*) + F(t,x,u - \varepsilon^*)\,. \end{aligned}$$

On the other hand, due to the fact that $F(\cdot,x,q)$ is non-decreasing for all $(x,q) \in \mathcal{M} \times \mathbb{R}$ we have that

$$\begin{aligned} \partial_t u^s &= D\Delta u^s + F(t+s,x,u^s) \\ &\geq D\Delta u^s + F(t,x,u^s)\,. \end{aligned}$$

Therefore, we can use the arguments in Proposition 2 to obtain that there exists $b(t,x) \in L^\infty(\mathbb{R} \times \mathcal{M})$ such that the function

$$v := u^s - (u - \varepsilon^*) \geq 0$$

satisfies the linear inequality

$$\partial_t v \geq D\Delta_{\mathcal{M}} v + b(t,x)v\,.$$

In consequence, applying Lemma 1 we obtain that there exists a constant $C_1 > 0$ such that

$$v(t_n - \tau, y_n) \leq C_1 v(t_n, y_n)\,.$$

On the one hand, if we recall that $y_n \in \overline{\Omega^-_{t_n-\tau}}$ and $d(y_n, \Gamma_{t_n-\tau})$ we obtain that

$$\begin{aligned} v(t_n - \tau, y_n) &= u^s(t_n - \tau, y_n) - u(t_n - \tau, y_n) + \varepsilon^* \\ &\geq (p^+ - \delta) - (p^- + \delta) + \varepsilon^* \\ &> \varepsilon^* > 0. \end{aligned}$$

But on the other hand we have that

$$\begin{array}{rcl} v(t_n, y_n) & = & u(t_n + s, y_n) - u(t_n, y_n) + \varepsilon^* \\ & < & -\varepsilon_n + \varepsilon^* \to 0\,. \end{array}$$

This contradiction implies that $\varepsilon^* = 0$, and the proof of the proposition is complete. □

Proposition 6 *$u^s \geq u$ in $\mathbb{R} \times \mathcal{M}$ for all $s \geq s_0$.*

Proof: Fix $s \geq s_0$. From Proposition 5 we have that $u^s(t, x) \geq u(t, x)$ for all $(t, x) \in \omega_A^-$, but the inequality is also valid when $(t, x) \in \mathbb{R} \times \mathcal{M} \setminus \omega_A^-$. Indeed, this can be proven using the same argument of Proposition 5, taking into that $F(t, x, \dot{)}$ is non-increasing in $[p^+ - \delta, +\infty)$ and that $u^s(t, x) \geq u(t, x)$ when $(t, x) \notin \omega_A^-$. □

Proposition 7 *Define*

$$s^* := \inf\{s > 0 \,:\, u^\sigma \geq u \quad \text{for all } \sigma \geq s\}\,.$$

Then $s^* = 0$.

Proof: By definition $0 \leq s^* \leq s_0$. Let us assume that $s^* > 0$ in order to reach a contradiction. Since $u^{s^*} \geq u$ in $\mathbb{R} \times \mathcal{M}$ we have two possibilities:

- **Case 1.** $\inf\{\, u^{s^*}(t, x) - u(t, x) \,:\, d(x, \Gamma_t) \leq A\} > 0$.

- **Case 2.** $\inf\{\, u^{s^*}(t, x) - u(t, x) \,:\, d(x, \Gamma_t) \leq A\} = 0$.

We will show that none of these two cases can hold.

Proof of Case 1.

By Lemma 3 it follows that $\partial_t u$ is globally bounded, which implies that there exists η_0 such that

$$u^{s^*-\eta}(t, x) \geq u(t, x) \text{ for all } \eta \in [0, \eta_0] \text{ and all } (t, x) \text{ satisfying } d(x, \Gamma_t) \leq A. \tag{16}$$

We claim that

$$u^{s^*-\eta} \geq u \quad \text{for all } \eta \in [0, \eta_0] \text{ and all } x \in \omega_A^-. \tag{17}$$

Indeed, let $x \in \overline{\Omega_t^-}$. If $d(x, \Gamma_t) = A$ then (16) implies that $u^{s^*-\eta}(t, x) \geq u(t, x)$, whereas if $d(x, \Gamma_t) > A$ then by Remark 2 it follows that $u(t, x) \geq p^- + \delta$. This implies that we can repeat the same arguments presented in Proposition 5 to prove that the claim holds.

On the other hand, by Remark 2 we also obtain that

$$u^{s^*}(t, x) \geq u(t, x) \geq p^- - \delta/2 \quad \text{if } x \in \overline{\Omega_t^+} \text{ and } d(x, \Gamma_t) \geq A.$$

In consequence, using the a priori bounds of Lemma 3 and decreasing η_0 if necessary, we have that

$$u^{s^*-\eta}(t, x) \geq p^- - \delta \quad \text{for all } \eta \in [0, \eta_0] \text{ and all } x \in \overline{\Omega_t^+} \text{ such that } d(x, \Gamma_t) \geq A. \tag{18}$$

In consequence, form (17), (16) and (18) it follows that

$$u^{s^*-\eta}(t,x) \geq u(t,x) \quad \text{for all } (t,x) \in \mathbb{R} \times \mathcal{M}.$$

Since this contradicts the minimality of s^*, Case 1 cannot hold.

Proof of Case 2.

By definition of the infimum, there exists a sequence $(t_n, x_n) \in \mathbb{R} \times \mathcal{M}$ such that

$$d(x_n, \Gamma_{t_n}) \leq A \text{ and } u^{s^*}(t_n, x_n) - u(t_n, x_n) \to 0 \text{ when } n \to +\infty.$$

Let us again remark that u^{s^*} is a supersolution of (1) and $u^{s^*} \geq u$. Therefore, applying Harnack inequality to the difference there exists $C_1 > 0$ such that

$$0 \leq u^{s^*}(t_n - s^*, x_n) - u(t_n - s^*, x_n) \leq C_1[u^{s^*}(t_n, x_n) - u(t_n, x_n)]\,.$$

Therefore, using $u^{s^*}(t_n - s^*, x_n) = u(t_n, s_n)$ it follows that

$$u(t_n, x_n) - u(t_n - s^*, x_n) \to 0 \text{ when } n \to +\infty.$$

Moreover, by induction we can show that

$$u(t_n, x_n) - u(t_n - ks^*, x_n) \to 0 \text{ when } n \to +\infty, \text{ for all } k \in \mathbb{N}.$$

Now fix $\varepsilon > 0$. By Definition 2 and Proposition 4 there exists $B_\varepsilon > 0$ such that

$$p^- < u(t,x) \leq p^- + \varepsilon \quad \text{for all } x \in \overline{\Omega_t^-} \text{ satisfying } d(x, \Gamma_t) \geq B_\varepsilon.$$

Since p^+ invades p^- then for all $s \leq t$ it follows that $\Omega_s^- \supset \Omega_t^+$, and also that $d(\Gamma_s, \Gamma_t) \to +\infty$ when $|t-s| \to +\infty$. These two properties and the boundedness of the sequence $d(x_n, \Gamma_{t_n})$ imply that there exists $m \in \mathbb{N}$ such that

$$x_n \in \overline{\Omega_{t_n - ms^*}^-} \text{ and } d(x_n, \Gamma_{t_n - ms^*}) \geq B_\varepsilon \text{ for all } n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$ we obtain that

$$p^- < u(t_n - ms^*, x_n) \leq p^- + \varepsilon\,.$$

In consequence,

$$u(t_n, x_n) - p^- \to 0 \text{ when } n \to +\infty. \tag{19}$$

From Definition 2 and Remark 2 there exist two positive real numbers B and $2\delta < p^+ - p^-$ such that

$$p^+ - \delta/2 \leq u(t,x) < p^+ \quad \text{for all } x \in \overline{\Omega_t^+} \text{ satisfying } d(x, \Gamma_t) \geq B.$$

Using Hypothesis (a) and the boundedness of the sequence $d(x_n, \Gamma_{t_n})$ we can construct a sequence $\tilde{x}_n$ such that

$$\tilde{x}_n \in \Gamma_{t_n - \tau} \text{ for all } n \in \mathbb{N} \text{ and } \sup\{d(x_n, \tilde{x}_n) : n \in \mathbb{N}\} < +\infty.$$

Moreover, by Hypothesis (b) there exist $r > 0$ and a sequence y_n such that

$$y_n \in \overline{\Omega_{t_n - \tau}^+},\ d(y_n, \tilde{x}_n) = r \text{ and } d(y_n, \Gamma_{t_n - \tau} \geq B \text{ for all } n \in \mathbb{N}.$$

In consequence, we have that

$$p^+ - \delta/2 \leq u(t_n - \tau, y_n) \ < p^+ \quad \text{for all } n \in \mathbb{N}.$$

Let us recall some important facts:

- $u(t,x)$ and p^- are both solutions of (1).
- $F(t,x,q)$ is locally Lipschitz and continuous in q, uniformly in (t,x).
- The sequence $d(x_n, y_n)$ is bounded.

Using these facts and Harnack inequality on the function

$$v := u - p^- > 0$$

we obtain that there is a $C_1 > 0$ such that

$$v(t_n - \tau, y_n) \leq C_1 v(t_n, x_n)\,.$$

On the one hand we know that $v(t_n, y_n) \to 0$ because of (19), but on the other hand we have that

$$\begin{array}{rcl} v(t_n-\tau, y_n) & = & u(t_n-\tau, y_n) - p^- \\ & \geq & p^+ - \delta/2 - p^- \\ & > & \delta/2 > 0\,. \end{array}$$

Thus we have reached a contradiction, which implies that Case 2 cannot hold. □

Proposition 8 *If $s > 0$ then $u^s(t,x) > u(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathcal{M}$.*

Proof: Choose any $s > 0$ and assume there is a point $(t_0, x_0) \in \mathbb{R} \times \mathcal{M}$ such that $u^s(t_0, x_0) = u(t_0, x_0)$. Since u^s is a supersolution of (1) and $u^s \geq u$, using the maximum principle we obtain that $u^s(t,x) = u(t,x)$ for all $(t,x) \in (-\infty, t_0] \times \mathcal{M}$.

Consider an arbitrary point $(t,x) \in (-\infty, t_0] \times \mathcal{M}$. Then for any $k \in \mathbb{N}$ we have that

$$0 \leq u(t,x) - p^- = u(t - ks, x) - p^-\,.$$

If we recall that $u(t,x)$ is a invasion of p^- by p^+ and take the limit $k \to +\infty$ it follows that $u(t,x) \equiv p^-$, which contradicts Proposition 4. □

This concludes the proof of Theorem 3.

4 Parabolic equations on Riemann manifolds

Let $\mathcal{L}$ be a second-order, negative-semidefinite, elliptic differential operator with continuous coefficients, whose expression in local coordinates is (using sum convention)

$$\mathcal{L}u = a_{ij}(t,\xi)\partial_{ij}u + b_i(t,\xi)\partial_i u + c(t,\xi)u\,.$$

We will also assume global boundedness of the coefficients of $\mathcal{L}$, i.e. the coefficients $a_{ij}(t,\xi)$, $b_i(t,\xi)$ and $c(t,\xi)$ have upper bounds that are independent of the chosen parametrization for the local coordinates.

Lemma 1 Harnack inequality
Let $T > 0$ and suppose that $u(t,x) \in C^2((0,T) \times \mathcal{M})$ solves

$$\begin{cases} \partial_t u - \mathcal{L}u \geq 0 & \text{in } (0,T) \times \mathcal{M}, \\ u \geq 0 & \text{in } (0,T) \times \mathcal{M}. \end{cases} \tag{20}$$

Let K be a compact subset of $\mathcal{M}$ and choose $\tau \in (0,T)$. Then for each $t \in (\tau, T)$ there exists a constant $C > 0$ (depending only on K, τ and the coefficients of $\mathcal{L}$) such that

$$\sup_K u(t-\tau, \cdot) \leq C \inf_K u(t, \cdot)\,. \tag{21}$$

Proof: Suppose there exists a parameterization $\phi : U \subset \mathbb{R}^n \to \mathcal{M}$ such that $K \subset \phi(U)$. Then we can consider the problem (20) in $(0,T) \times U$, and hence we can use Harnack's inequality in $\mathbb{R}^n$ to deduce that there exists $C > 0$ such that the inequality (21) holds (see Evans [5], Theorem 10, p. 370 and Lieberman [9], Theorem 6.27, p. 129).

For a general K, take a compact finite cover $K_1, \ldots, K_d$ such that for each $i = 1, \ldots, d$ there exists a chart $\phi_i : U_i \subset \mathbb{R}^n \to \mathcal{M}$ satisfying $K_i \subset \phi_i(U_i)$. Now we apply the previous argument to each K_i find a constant C_i $(i = 1, \ldots, d)$ for which inequality (21) holds in K_i. Choosing $C = \max\{C_1, \ldots, C_d\}$ we obtain the result. $\square$

Let $p^- < u(t,x) < p^+$ for all $(t,x) \in \mathbb{R} \times \mathcal{M}$, and suppose $F(t,x,u)$ satisfies the following conditions:

- $s \mapsto F(s,x,u)$ is nonincreasing for all $s \in \mathbb{R}$.
- There exists $\delta > 0$ such that $q \mapsto F(t,x,q)$ is nonincreasing for all $q \in \mathbb{R} \setminus (p^- + \delta, p^+ - \delta)$.

Lemma 2 Strong maximum principle
Let $u(t,x)$ and $p(t,x)$ two solutions of

$$\partial_t u = \mathcal{L}u + F(t,x,u) \quad \text{in } \mathbb{R} \times \mathcal{M} \tag{22}$$

such that $u(t,x) \geq p(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathcal{M}$. If there is a point $(t_0, x_0) \in \mathbb{R} \times \mathcal{M}$ such that $u(t_0,x_0) = p(t_0,x_0)$ then $u(t,x) = p(t,x)$ for all $(t,x) \in [0,t_0] \times \mathcal{M}$.

Proof: Let $(t_0,x_0) \in \mathbb{R} \times \mathcal{M}$ be such that $u(t_0,x_0) = p(t_0,x_0)$. Using a parameterization $\phi : \bar{U} \subset \mathbb{R}^n \to \mathcal{M}$ with $(t_0,x_0) \in \phi(U)$, we can consider equation (22) and the solutions $u(t,x)$, p(t,x) in $U \subset \mathbb{R}^n$, a smooth domain containing $(t_0,y_0) = \phi^{-1}(t_0,x_0)$.

By hypothesis we have $u(t,x) \geq p(t,x)$ for all $(t,y) \in \mathbb{R} \times \bar{U}$ and $u(t_0,y_0) = p(t_0,y_0)$. Therefore, using the strong maximum principle in $\mathbb{R}^n$ it follows that $u(t,y) = p(t,y)$ for all $(t,y) \in [0,t_0] \times \bar{U}$.

Now define the set

$$C = \{\,(t,x) \in \mathbb{R} \times \mathcal{M} \,:\, u(t,x) = p(t,x)\,\}\,.$$

C is closed because $u(t,x)$ and $p(t,x)$ are continuous, and we have just shown that it is also open. In consequence, since $\mathcal{M}$ is connected, $C = [0,t_0] \times \mathcal{M}$. $\square$

Remark 3 *If the solution of equation (22) is unique then Lemma 2 holds for all times:*

$$u(t,x) = p(t,x) \quad \text{for all } \ (t,x) \in \mathbb{R} \times \mathcal{M}.$$

Lemma 3 A priori estimates
If $u(t,x) \in C^2(\mathbb{R} \times \mathcal{M})$ is solution of the equation

$$\partial_t u = \mathcal{L}u + F(t,x,u) \quad \text{in } \mathbb{R} \times \mathcal{M}$$

and F is bounded then $\partial_t u$ and $\nabla_x u$ are globally bounded.

Proof: Using the hypothesis of global boundedness of the coefficients of $\mathcal{L}$ we can assume that $\mathcal{M}$ is an open subset of $\mathbb{R}^N$. For Euclidean domains there are well-known a priori estimates for parabolic equations, namely, if the coefficients are in L^p then the solution is in L^p as well. For a proof of this general fact see e.g. Ladyženskaja *et al* [8] (Chapters 1 and 4) and Solonnikov [11]. $\square$

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